

## Description

This invention relates to parallel computation circuits.

In spite of the tremendous computational power of digital computers, for a variety of reasons many practical problems cannot effectively be solved by the digital computer. The reason may be that a closed form solution is unavailable for the particular formulation of the problem and, therefore, numerical methods must be employed; or it may be that the number of variables in an optimization problem is large, and finding the optimum solution from the very large set of possible solutions is just too great a task to be performed by a digital computer in a reasonable time.

Heretofore, most artisans have either suffered the limitations of general purpose digital computers, or developed special purpose digital computers to solve their particular problems more efficiently. Recently, advances have brought to the forefront a class of highly parallel computation circuits that solve a large class of complex problems in analog fashion. These circuits comprise a plurality of amplifiers having a sigmoid transfer function and a resistive feedback network that connects the output of each amplifier to the input of the other amplifiers. Each amplifier input also includes a capacitor connected to ground and a resistor connected to ground. Input currents are fed into each amplifier input, and output is obtained from the collection of output voltages of the amplifiers.

Biological Cybernetics, vol. 52, no. 3, July 1985, pages 141-152, Springer Verlag, Berlin, DE; J.J. Hopfield et al. "Neural" computation of decisions in optimisation problems" discusses the use of the now-classic feedback network as a minimization solver. As an example of such a use Hopfield chose to solve the "traveling salesman problem" (TSP). The solution of the TSP is discrete. City A is a position  $i$  on the tour, City B is at position  $d$  in the tour, etc. To specify the position of a city in the tour, a multi-bit set of outputs was employed. Having chosen  $M$  bits to specify each city, and having chosen  $N$  cities for their problem, the network was necessarily sized at  $MN$  outputs,  $MN$  amplifiers, and  $MN \times MN$  feedback matrix. Having established the general framework, an appropriate energy function that will tend to give a desired solution is reflected in equations 8 and 9 on page 146. The network was then simulated on a digital computer (see page 147).

The invention will be described with referenced to the accompanying drawings, in which:

FIG. 1 shows a known highly interconnected analog network; and

FIG. 2 shows a network embodying the invention.

FIG. 1 depicts a generalised circuit having five amplifiers (10-14) with positive and negative outputs  $V_1$  and  $-V_1$  on line pair 21,  $V_2$  and  $-V_2$  on line pair 22,  $V_3$  and  $-V_3$  on line pair 23,  $V_4$  and  $-V_4$  on line pair 24, and  $V_N$  and  $-V_N$  on line pair 25. Those outputs are connected to an interconnection block 20 which has output lines 41-45 connected to the input ports of amplifiers 10-14, respectively. Within interconnection block 20, each output voltage  $V_i$  is connected to each and every output line of block 20 through a conductance (e.g., resistor). For convenience, the conductance may be identified by the specific output line (i.e., source) that is connected by the conductance to a specific voltage line. For example,  $T_{21}^+$  identifies the conductance that connects the positive output  $V_2$  of amplifier 11 to the input of the first amplifier (line 41).

Also connected to each amplifier input port is a resistor and a capacitor whose second lead is grounded, and means for injecting a current (from some outside source) into each input port.

Applying Kirchoff's current law to the input port of each amplifier  $i$  of FIG. 1 yields the equation:

$$C_i (du_i/dt) = -u_i/R_i + \sum_j (T_{ij}^+ - T_{ij}^-) V_j + I_i \quad (1)$$

where

$C_i$  is the capacitance between the input of amplifier  $i$  and ground,

$$\frac{1}{R_i}$$

is the equivalent resistance and it equals

$$\frac{1}{p_i} + \sum_j \frac{1}{R_{ij}}$$

where  $p_i$  is the resistance between the input of amplifier  $i$  and ground,

$u_i$  is the voltage at the input of amplifier  $i$ ,

$T_{ij}^+$  is the a conductance between the positive output of amplifier  $j$  and the input of amplifier  $i$ ,

$T_{ij}^-$  is the a conductance between the negative output of amplifier  $j$  and the input of amplifier  $i$ ,

$V_j$  is the positive output voltage of amplifier  $j$ , and

$I_i$  is the current driven into the input port of amplifier  $i$  by an external source.

When  $T_{ij}^+$  and  $T_{ij}^-$  are disjoint,  $T_{ij}^+ - T_{ij}^-$  may for convenience be expressed as  $T_{ij}$ , and it is well known that a circuit satisfying Equation (1) with symmetric  $T_{ij}$  terms is stable. It is also well known that such a circuit responds to applied stimuli, and reaches a steady state condition after a short transition time. At steady state,  $du/dt=0$  and  $dV/dt=0$ .

With this known stability in mind, the behavior of other functions may be studied which relate to the circuit of FIG.1 and involve the input signals of the circuit, the output signals of the circuit, and/or the circuit's internal parameters.

For example, a function was studied that has the form

$$E = -\frac{1}{2} \sum_{i,j} T_{ij} V_i V_j + \sum_i (1/R_i) \int_0^{V_i} g_i^{-1}(v) dv - \sum_i I_i V_i \quad (2)$$

It is observed that the integral of the function  $g_i^{-1}(V)$  approaches 0 as the gain of amplifier  $i$  approaches infinity. Also, the time derivative of the function  $E$  is negative, and that it reaches 0 when the time derivative of voltages  $V_i$  reaches 0. Since equation (1) assures the condition of  $dV/dt$  approaching 0 for all  $i$ , the function  $E$  of equation (2) is assured of reaching a stable state. The discovery of this function  $E$  led to the use of the FIG.1 circuit in problem solving applications in associative memory applications, and in decomposition problems.

The FIG. 1 circuit can solve the above classes of problems when those problems are structured so that the minimization of a function having at most second order terms in some parameters of the problem results in the desired solution. Other problems, however, may require the minimization of equations that contain terms of order higher than two. Those may be problems that perhaps can otherwise be stated with second order terms, but the statement with higher order terms is more meaningful, or they may be problems that can only be described with the use of higher order terms.

The invention seeks to provide a parallel computation circuit for solving problems characterised by higher order terms.

According to this invention there is provided a parallel circuit as claimed in claim 1.

An embodiment of the invention comprises an interconnected analog network that is constructed with inter-neurons that account for terms of order greater than two. The network comprises analog amplifiers that are connected with a resistive interconnection matrix which, like the prior art network, connects each amplifier output to the input of all other amplifiers. The connections embodied in the matrix are achieved with conductances whose values are computed in accordance with the set of problem restrictions, while the cost variables of the problem which are to be minimized are applied as input signals.

The invention may be more fully understood and appreciated when applied to a specific problem and, to that end, the following detailed description presents the best mode for constructing a highly parallel computation circuit for solving a minimization problem characterised by an equation that has terms of order greater than two.

For purposes of illustration, the problem selected is the classic routing problem, such as encountered in the airline industry. In such a problem there may be  $N$  points where stops can be made (e.g., cities), and there is a cost associated with travelling from one point to the next. The object is to find a path in conformance to the rules of the problem at hand which connects a given origin point with a specific destination point at a minimum attendant overall cost.

One way to solve this problem is to construct a cost function that describes the desired attributes of the final solution, and to select the variables of that function so that the cost is minimized. Equation (3) below is one such cost function

$$E = \sum_{AB} \sum_{AB} c_{AB} V_{AB} \quad (3)$$

where  $V_{AB}$  represents a link from point (city) A to point (city) B, and  $c_{AB}$  represents the direct cost associated with taking link AB. The presence or absence of link AB in the solution is indicated by  $V_{AB}$  either taking on the value 1 or 0, respectively, and thus, the terms  $c_{AB} V_{AB}$  combine to represent the overall direct cost of taking a particular route. A specification that a particular link may not be present in the solution tour is accomplished by assigning a high cost factor  $c_{AB}$  to the impermissible link.

In a routing problem of N points, each point may have a link to all other points and, hence, at most (N) (N-1) links can exist. This number corresponds to the number of terms in Equation (3) and is equal to the number of direct cost terms in Hopfield's model. It may be noted that since it is links that are considered rather than cities, the V terms herein are characterized by two subscripts rather than one. Consequently, we might expect that terms which otherwise correspond to the  $T_{ij}$  terms found in Equation (2) would appear with four subscripts, e.g.,  $T_{ijkl}$ . Also, the summations performed would be over four indices rather than two.

Although the Equation (3) cost function describes the direct cost of a tour, in itself it is not sufficient to insure that a solution would be found to the tour minimization problem. For example, Equation (3) clearly permits a tour specification of a link DE followed by a link DF, but links DE and DF emanate from the same point (D) and cannot follow each other. This is an "outgoing branch" condition. Since the "outgoing branch" condition is unacceptable in the final solution, we discourage it by adding the penalty function

$$E_1 = \lambda_1 \sum_{AB} \sum_{AB} V_{AB} \sum_C V_{AC} \quad (4)$$

to the overall cost function of Equation (3). Equation (3) says, in essence, that a traversal from A to B followed by a traversal from A to B shall be very costly ( $\lambda$  is large).

A corollary to the "outgoing branch" condition is the "incoming branch" condition, where more than one link is chosen to reach a certain point. This unwanted condition is discouraged by adding the penalty function

$$E_2 = \lambda_2 \sum_{AB} \sum_{AB} V_{AB} \sum_C V_{CB} \quad (5)$$

to the overall cost function of Equation (3).

The terms contributed by Equations (4) and (5) are similar to the first term in Equation (2), in that the variable V appears in the second degree, and so those terms present no problem in the realization of a circuit to minimize the cost function resulting from combining Equations (3), (4), and (5). However, preventing the "outgoing branch" and "incoming branch" conditions may not be sufficient.

A further condition of Equation (3) relates to the selection of some links and the non-selection of some other possible links. When one visualizes the physical problem, it is readily apparent that having taken one link to reach a point A, some link emanating from point A must also be taken; otherwise the tour cannot properly be completed. Stated in other words, having taken a certain link and having taken one of a number of other links would not comport with a valid solution to the physical problem. Conversely, a solution where a link is taken from a point A but no link is taken to reach point A also does not comport with a physically realizable solution to the problem. This continuity requirement applied to all points of the tour, save for the starting (origin) point,  $\alpha$ , and the end (destination) point,  $\omega$ . At those points we wish not to discourage the breaking of continuity, and in fact we wish to encourage it. Accordingly, an additional function is constructed and added to the overall cost function of Equation (3), which considers each link AB and imposes penalties for certain conditions at points A and B. That function (discussed in detail below) is

$$\begin{aligned}
 E_3 = & \lambda_3 \sum_{AB \neq \omega} \sum V_{AB} \Pi_{BC} + \lambda_4 \sum_A \sum V_{A\omega} + \lambda_5 \sum_C \sum V_{\omega C} \\
 & + \lambda_6 \sum_{A \neq \alpha} \sum V_{AB} \Pi_{CA} + \lambda_7 \sum_B \sum V_{\alpha B} + \lambda_8 \sum_C \sum V_{C\alpha}
 \end{aligned} \quad (6)$$

where the first three terms concern themselves with continuity at point B, and the last three terms concern themselves with continuity at point A.

With respect to Point B, as long as it is not the end point,  $\omega$ , the desired solution must include a link from point B to some point C. Accordingly, the first term in Equation (6) imposes a penalty for not including a link from point B to some point C. Stated in other words, a penalty  $\lambda_3$  is imposed when a link AB is taken AND a link BC (where C is any other point) is not taken. The product operator  $\Pi$  performs the logic AND function and, hence, when a link BC is taken, the resulting product is zero and penalty  $\lambda_3$  is not imposed. When B is the destination point, however, the first term imposes no penalties, but the second term imposes a penalty for not entering  $\omega$ , and the third term imposes a penalty for exiting  $\omega$ .

With respect to point A, as long as it is not the starting point,  $\alpha$ , the desired solution must include a link from some point C to point A. Accordingly, the fourth term in Equation (6) imposes a penalty for not including a link from some point C to point A. When point A is the starting point, however, the fourth term imposes no penalties, but the fifth term imposes a penalty for not exiting  $\alpha$ , and the sixth term imposes a penalty for entering  $\alpha$ .

There are two situations that may occur where continuity is maintained but it is undesirable. These are when a "stand still" path is taken, such as  $V_{AA}$ , and when a "backtrack" path is taken, such as  $V_{AB}$  followed by  $V_{BA}$ . These undesirable situations are penalized with function  $E_4$  that has the form

$$E_4 = \lambda_9 \sum_A V_{AA} + \lambda_{10} \sum_{AB} V_{AB} V_{BA} \quad (7)$$

Utilizing the equality

$$\sum_{A \neq B} V_{AC} = \sum_A V_{AC} - V_{BC} \quad (8)$$

the cost function which combines Equations (3) through (7) equals to the following:

$$\begin{aligned}
 E = & \sum_{AB} [c_{AB} V_{AB} + \lambda_1 \sum_C V_{AB} V_{AC} + \lambda_2 \sum_C V_{AB} V_{CB}] \\
 & + \sum_{AB} [\lambda_3 V_{AB} \Pi_{BC} + \lambda_6 V_{AB} \Pi_{CA}] \\
 & - \sum_A \lambda_3 V_{A\omega} \Pi_{\omega C} - \sum_B \lambda_6 V_{\alpha B} \Pi_{C\alpha}
 \end{aligned}$$

$$\begin{aligned}
& + \lambda_4 \overline{\sum_A V_{Au}} + \lambda_5 \sum_C V_{uC} \\
& + \lambda_7 \overline{\sum_B V_{\alpha B}} + \lambda_8 \sum_V \sum_{\alpha C} V_{\alpha C} \\
& + \lambda_9 \sum_A V_{AA} + \lambda_{10} \sum_{AB} \sum_{AB} V_{AB} V_{BA} \quad (9)
\end{aligned}$$

Equation (9) represents a valid and useful statement of the overall cost function of the routing problem. Unfortunately, the terms having the product operator  $\Pi$  are approximately in the  $N^{\text{th}}$  degree of the variable  $V$ . The overall cost function is therefore not of the form described in the aforementioned copending applications, and it would appear that this overall cost function cannot be mapped onto the circuit of FIG. 1 as described by the prior art. We discovered a way to restructure the problem and make it solvable with a circuit akin to the circuit of FIG. 1.

In solving the minimization problem in accordance with Equation (2), the FIG. 1 circuit develops an output voltage for each  $V$  term and, when the circuit reaches steady state, these voltages assume either the value 1 or 0. Realizing that the product terms (e.g.,  $\prod_C V_{BC}$ ) are also restricted in the steady state to 0 or 1, we replace terms of the form  $\prod_C V_{BC}$  with single variables of the form  $V_{\text{Bout}}$  and the term  $\prod_C V_{CA}$  with the single variable  $V_{\text{InA}}$ . This yields a cost function equation that has terms with  $V$  variables only in the first or second degree. On first blush this process of replacement appears to be merely form over substance; but that is not the case, as shown below.

We call the  $V_{\text{Bout}}$  and  $V_{\text{InA}}$  terms inter-neurons which, in their steady state, are either 1 (turned on) or 0 (turned off). Whatever state the inter-neurons assume, however, they must satisfy their definitional equality; and this imperative may be incorporated in the overall cost function by assigning a penalty for failing to satisfy the equality. Employing De Morgan's theorem offers the identity

$$\prod_Y V_{XY} = \overline{\sum_Y \overline{V_{XY}}}$$

and in our application where the final state of an inter-neuron is either 1 or 0,

$$\sum_Y \overline{V_{XY}}$$

is also equal to

$$1 - \sum_Y V_{XY}.$$

Employing the above, an additional cost function to be included in the overall cost function can be created which introduces the concept of a penalty for failure to satisfy the definitional requirement of inter-neurons. Specifically, the term

$$\prod_C V_{BC}$$

in Equation (9) is replaced with  $V_{\text{Bout}}$  and the cost term

$$\lambda_{11} (V_{\text{Bout}} + \sum_C V_{BC} - 1)^2$$

is added. Similarly, the term

$$\overline{\sum_C V_{CA}}$$

in Equation (9) is replaced with  $V_{inA}$  and the cost term

$$\lambda_{12} (V_{inA} + \sum_C V_{CA} - 1)^2$$

is added. Finally, the term

$$\overline{\sum_A V_{Au}}$$

is replaced with

$$1 - \sum_A V_{Au}$$

and the term

$$\overline{\sum_B V_{qB}}$$

is replaced with

$$1 - \sum_B V_{qB}$$

Combining the above cost functions yields the final overall cost function:

$$\begin{aligned} E = & \sum_{AB} (c_{AB} V_{AB} + \lambda_1 \sum_C V_{AB} V_{AC} + \lambda_2 \sum_C V_{AB} V_{CB}) \\ & + \sum_{AB} (\lambda_3 V_{AB} V_{Bout} + \lambda_6 V_{AB} V_{inA}) \\ & - \sum_A \lambda_3 V_{Au} V_{uout} - \sum_B \lambda_6 V_{qB} V_{ind} \\ & + \lambda_4 (1 - \sum_A V_{Au}) + \lambda_5 \sum_C V_{uC} \\ & + \lambda_7 (1 - \sum_B V_{qB}) + \lambda_8 \sum_C V_{qC} \\ & + \lambda_9 \sum_A V_{AA} + \lambda_{10} \sum_{AB} V_{AB} V_{BA} \\ & + \lambda_{11} \sum_B (V_{Bout} + \sum_C V_{BC} - 1)^2 + \lambda_{12} \sum_A (V_{inA} + \sum_C V_{CA} - 1)^2. \quad (10) \end{aligned}$$

Equation (10) has the general form of Equation (2) in that it has terms in V only of the first degree and of the second degree. It differs from Equation (2) only in that the terms don't all appear as summations over four indices. That can be remedied, however, in a straightforward manner. For example, the term

$$\sum_{ABC} \lambda_i v_{AB} v_{AC}$$

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can be converted to

$$\sum_{ijkl} \lambda_l \delta_k^i v_{ij} v_{kl}$$

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(or rewritten in a simpler form as

$$\sum_{i,j,k,l} \lambda_l \delta_k^i v_{ij} v_{kl}),$$

where  $\delta_k^i$  is a function whose value is 1 when  $k=i$ , and 0 otherwise. In addition to the introduction of the  $\delta$  function, the above makes a simple variable transformation to reduce confusion.

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The above procedure can be applied to each and every term in Equation (10), yielding the table:

TABLE

|    |                                                                     |               |                                                                              |
|----|---------------------------------------------------------------------|---------------|------------------------------------------------------------------------------|
| 5  | $\sum_{A,B} C_{AB} V_{AB}$                                          | $\rightarrow$ | $\sum_{i,j} C_{ij} V_{ij}$                                                   |
|    | $\sum_{A,B,C} \lambda_1 V_{AB} V_{AC}$                              | $\rightarrow$ | $\sum_{i,j,k,l} \lambda_1 \delta_{kl}^i V_{ij} V_{kl}$                       |
| 10 | $\sum_{A,B,C} \lambda_2 V_{AB} V_{CB}$                              | $\rightarrow$ | $\sum_{i,j,k,l} \lambda_2 \delta_{kl}^j V_{ij} V_{kl}$                       |
|    | $\sum_{A,B} \lambda_3 V_{AB} V_{Cout}$                              | $\rightarrow$ | $\sum_{i,j,k,l} \lambda_3 \delta_{kl}^j \delta_{j1}^{out} V_{ij} V_{kl}$     |
|    | $\sum_A (-\lambda_3) V_{Au} V_{uout}$                               | $\rightarrow$ | $\sum_{i,j,k,l} (-\lambda_3) \delta_{kl}^j \delta_{j1}^{uout} V_{ij} V_{kl}$ |
| 15 | $\sum_{A,B} \lambda_6 V_{AB} V_{inA}$                               | $\rightarrow$ | $\sum_{i,j,k,l} \lambda_6 \delta_{kl}^i \delta_{i1}^{in} V_{ij} V_{kl}$      |
|    | $\sum_B (-\lambda_6) V_{dB} V_{ind}$                                | $\rightarrow$ | $\sum_{i,j,k,l} (-\lambda_6) \delta_{kl}^i \delta_{i1}^{ind} V_{ij} V_{kl}$  |
|    | $\lambda_4$                                                         | $\rightarrow$ | $\lambda_4$                                                                  |
| 20 | $\sum_A (-\lambda_4) V_{Au}$                                        | $\rightarrow$ | $\sum_{i,j} (-\lambda_4) \delta_{j1}^u V_{ij}$                               |
|    | $\sum_C \lambda_5 V_{uC}$                                           | $\rightarrow$ | $\sum_{i,j} \lambda_5 \delta_{i1}^u V_{ij}$                                  |
| 25 | $\lambda_7$                                                         | $\rightarrow$ | $\lambda_7$                                                                  |
|    | $\sum_B (-\lambda_7) V_{dB}$                                        | $\rightarrow$ | $\sum_{i,j} (-\lambda_7) \delta_{i1}^d V_{ij}$                               |
|    | $\sum_C \lambda_8 V_{Cd}$                                           | $\rightarrow$ | $\sum_{i,j} \lambda_8 \delta_{j1}^d V_{ij}$                                  |
| 30 | $\sum_A \lambda_9 V_{AA}$                                           | $\rightarrow$ | $\sum_{i,j} \lambda_9 \delta_{j1}^i V_{i,j}$                                 |
|    | $\sum_{AB} \lambda_{10} V_{AB} V_{BA}$                              | $\rightarrow$ | $\sum_{i,j,k,l} \lambda_{10} \delta_{kl}^j \delta_{l1}^i V_{ij} V_{kl}$      |
| 35 | $\sum_B \lambda_{11} \left  (V_{Bout}-1) + \sum_C V_{BC} \right ^2$ |               |                                                                              |
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$$\begin{aligned}
& \sum_B \lambda_{11} V_{Bout} V_{Bout} \rightarrow \sum_{i,j,k,l} \lambda_{11} \delta_{kl}^{i6j6out} v_{ij} v_k \\
& \sum_B \lambda_{11} (-2) V_{Bout} \rightarrow \sum_{i,j} (-2\lambda_{11}) \delta_{ij}^{out} v_{i,j} \\
& \sum_B \lambda_{11} \rightarrow \sum \lambda_{11} \delta_j^i \\
& \sum_{B,C} 2\lambda_{11} V_{Bout} V_{BC} \rightarrow \sum_{i,j,k,l} 2\lambda_{11} \delta_{kl}^{i6j6out} v_{i,j} v_k \\
& \sum_{BC} (-2\lambda_{11}) V_{BC} \rightarrow \sum_{i,j} (-2\lambda_{11}) v_{i,j} \\
& \sum_{BCC'} \lambda_{11} V_{BC} V_{BC'} \rightarrow \sum_{i,j,k,l} \lambda_{11} \delta_{kl}^{i6j6out} v_{i,j} v_{k,l} \\
& \sum_A \lambda_{12} \left| (V_{inA} - 1) + \sum_C V_{CA} \right|^2 \\
& \sum_A \lambda_{12} V_{inA} V_{inA} \rightarrow \sum \lambda_{12} \delta_i^{in6j6out} v_{i,j} v_{k,l} \\
& \sum_A -2\lambda_{12} V_{inA} \rightarrow -2\lambda_{12} \delta_i^{in} v_{i,j} \\
& \sum_A + \lambda_{12} \rightarrow \sum \lambda_{12} \\
& \sum_{A_1C} 2\lambda_{12} V_{inA} V_{CA} \rightarrow \sum_{i,j,k,l} 2\lambda_{12} \delta_{kl}^{in6j6out} v_{i,j} v_k \\
& \sum_{AC} -2\lambda_{12} V_{CA} \rightarrow \sum_{i,j} -2\lambda_{12} v_{i,j} \\
& \sum_{ACC'} \lambda_{12} V_{CA} V_{C'A} \rightarrow \sum_{i,j,k,l} \lambda_{12} \delta_{kl}^{in6j6out} v_{i,j} v_{k,l}
\end{aligned}$$

which may be written as

$$E = \sum_{i,j} I_{ij} v_{ij} + \sum_{i,j,k,l} T_{ijkl} v_{ij} v_{kl} + K \quad (11)$$

where

$$\begin{aligned}
I_{ij} &= | c_{ij} - \lambda_4 \delta_j^w + \lambda_5 \delta_i^w - \lambda_7 \delta_i^d + \\
& \lambda_8 \delta_j^d + \lambda_9 \delta_j^i - 2\lambda_{11} \delta_j^{out} - 2\lambda_{11} - 2\lambda_{11} \delta_i^{in} - 2\lambda_{12} |, \\
T_{ijkl} &= | \lambda_1 \delta_k^i + \lambda_2 \delta_l^j + \lambda_3 \delta_k^j \delta_l^{out} - \lambda_3 \delta_k^j \delta_l^{out} \delta_j^w + \lambda_6 \delta_l^i \delta_k^{in} - \\
& \lambda_6 \delta_l^i \delta_k^{in} \delta_i^d + \lambda_{10} \delta_k^j \delta_l^i + \lambda_{11} \delta_k^j \delta_l^{out} + 2\lambda_{11} \delta_k^i \delta_l^{out} + \\
& \lambda_{11} \delta_k^i + \lambda_{12} \delta_i^{in} \delta_k^j \delta_l^i + 2\lambda_{12} \delta_i^{in} \delta_l^j + \lambda_{12} \delta_l^j |,
\end{aligned}$$

and

$$K = \lambda_7 + \sum_i (\lambda_{11} + \lambda_j^i).$$

A perusal of Equations (10) and (11) reveals that the terms corresponding to the  $I_{ij}$  coefficients contain the direct cost factors ( $c_{ij}$ ) of the problem, and  $\lambda$  and  $\delta$  terms. The  $T_{ijkl}$  coefficients comprise only  $\lambda$  and  $\delta$

terms. The  $\delta$  and  $\lambda$  terms define the ever-present restrictions on valid solutions for the type of problem described, and hence the mapping of those terms onto the  $T_{ij}$  elements of Equation (2) leads to the advantageous result that a highly parallel computation circuit can be constructed with feedback terms  $T_{ij}$  being independent of the specific routing problem to be solved.

- 5 FIG. 2 depicts a circuit that attains a minimum for the cost function of Equations (10) and (11). It is similar to the FIG. 1 circuit in that there are output voltages

$$v_1, v_2, \dots v_{N^2}$$

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(obtained from corresponding amplifiers) and an interconnection block 30 that connects those output voltages to the various input ports of the amplifiers. It differs, however, in that additional amplifiers, 16, 17, 18 are provided that develop the inter-neuron outputs

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$$v_{N^2+1}, v_{N^2+2}, \dots v_{N^2+N'}$$

- 20 and those, too, are connected to interconnection block 30. In this manner the inter-neurons affect the output voltages, and the output voltages affect the inter-neurons to insure that the inter-neurons follow their definitional equality, causing the overall cost function (10) to be minimized. FIG. 2 also includes a network 40 which implements the connections embodied in the coefficient of the  $I_{ij}$  term of Equation (11). Evaluation of the  $I_{ij}$  and  $T_{ijkl}$  terms of Equation (11) for a particular routing problem yields the actual conductance values in the feedback matrices of FIG. 2.

- 25 The above describes a very specific example of a system where the energy to be minimized is expressed in terms of output signal products of degree greater than two. Because of the specific nature of the illustrative example, the resulting equations (10 and 11) are rather long and somewhat complex. So as not to overlook the forest for the trees, it is reiterated here that our concept of inter-neurons is applicable whenever the energy function has a term that contains a product such as

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$$v_{AB} \prod_C v_{CA} \text{ or } v_{AB} \prod_D v_{CD}.$$

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The first of these terms converts to  $V_{AB}V_{inA}$  and a penalty term of the form

$$\lambda (v_{inA} + \sum_C v_{CA} - 1)^2$$

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is added to the energy function. Similarly, the second of these terms converts to  $V_{AB}V_{out}$  and the penalty of the form

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$$\lambda (v_{out} + \sum_D v_{CD} - 1)^2$$

- 50 is added to the energy function. Introduction of the  $V_{inA}$  and  $V_{out}$  terms corresponds to the inclusion of an input inter-neuron and an output inter-neuron, respectively; and that translates to an additional active element in the system that implements the energy minimization process. The feedback matrix that connects the output of each amplifier to the input of all amplifiers, expressed as

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$$\sum_{i,j,k,l} T_{ijkl} v_{ij} v_{kl},$$

is augmented for each input inter-neuron with a term of the form

$$\sum_{i,j,k,l} \left| \delta_k^i \delta_l^{in} + \lambda \delta_k^i \delta_l^{in} \delta_k^j \delta_l^j + 2\lambda \delta_k^i \delta_l^{in} \delta_k^j + \lambda \delta_l^j \right|$$

and is augmented for each output inter-neuron with a term of the form

$$\sum_{i,j,k,l} \left| \delta_k^j \delta_l^{out} + \lambda \delta_k^j \delta_l^i \delta_k^i + 2\lambda \delta_k^j \delta_l^i + \lambda \delta_k^i \right|.$$

The matrix that provides input currents to the amplifiers is augmented in a similar fashion as shown in Equation (11).

Of course other index patterns in the higher order product terms of the energy function would yield somewhat different augmentation terms to the  $V_{ijkl}$  matrix and to the  $I_{ij}$  matrix; but those variations are clearly within the contemplation of our invention.

Expressing the invention in even broader terms, whenever the energy function has terms of order greater than two, such as

$$v_u \cdot v_w \cdot v_x \cdot v_y \cdot v_z \dots,$$

such terms are converted to

$$v_u \cdot v_m,$$

where each  $V_m$  is a new inter-neuron term, and a penalty term of the form

$$\lambda \{ v_m + (-1 + \overline{v_w} + \overline{v_x} + \overline{v_y} + v_z + \dots) \}^2$$

is added to the energy function. Introduction of the  $V_m$  translates to an additional active element in the system that implements the energy minimization process. The feedback matrix that connects the output of each amplifier to the input of all amplifiers is augmented for each inter-neuron by the coefficients of the equation

$$v_u v_m + \lambda \{ v_m + (-1 + \overline{v_w} + \overline{v_x} + \overline{v_y} + v_z + \dots) \}^2.$$

## Claims

1. A parallel circuit for developing N simultaneous output signals which minimize an energy function that is expressed by said output signals in terms of products of different ones of said output signals, said parallel circuit comprising a plurality of amplifiers and a feedback conductance matrix connecting the output of each of said amplifiers,  $V_{ij}$ , to the input of each of said amplifiers, with each of said connections being effected through a conductance  $T_{ijkl}$ ; where said feedback matrix comprises conductances  $T_{ijkl}$  connecting the output of amplifier  $ij$  to the input of amplifier  $kl$  with conductance values which are linear combinations of  $\lambda_x$  and  $\delta_z^y$  terms, where  $\lambda_x$  are predetermined penalty constants and  $\delta_z^y$  is a function whose value is one when  $y=z$ , and zero otherwise, where  $y$  and  $z$  belong to the set



is augmented for each input inter-neuron with a term of the form

$$\sum_{i,j,k,l} \left| \delta_l^i \delta_k^{in} + \lambda \delta_l^{in} \delta_k^i \delta_l^j + 2\lambda \delta_l^{in} \delta_l^j + \lambda \delta_l^j \right|$$

and is augmented for each output inter-neuron with a term of the form

$$\sum_{i,j,k,l} \left| \delta_k^j \delta_l^{out} + \lambda \delta_k^j \delta_l^i \delta_j^i + 2\lambda \delta_k^i \delta_l^i + \lambda \delta_k^i \right|.$$

The matrix that provides input currents to the amplifiers is augmented in a similar fashion as shown in Equation (11).

Of course other index patterns in the higher order product terms of the energy function would yield somewhat different augmentation terms to the  $V_{ijkl}$  matrix and to the  $I_{ij}$  matrix; but those variations are clearly within the contemplation of our invention.

Expressing the invention in even broader terms, whenever the energy function has terms of order greater than two, such as

$$v_u \cdot v_w \cdot v_x \cdot v_y \cdot v_z \dots$$

such terms are converted to

$$v_u \cdot v_m,$$

where each  $V_m$  is a new inter-neuron term, and a penalty term of the form

$$\lambda [v_m + (-1 + \overline{v_w} + \overline{v_x} + \overline{v_y} + v_z + \dots)]^2$$

is added to the energy function. Introduction of the  $V_m$  translates to an additional active element in the system that implements the energy minimization process. The feedback matrix that connects the output of each amplifier to the input of all amplifiers is augmented for each inter-neuron by the coefficients of the equation

$$v_u v_m + \lambda [v_m + (-1 + \overline{v_w} + \overline{v_x} + \overline{v_y} + v_z + \dots)]^2.$$

## Claims

1. A parallel circuit for developing N simultaneous output signals which minimize an energy function that is expressed by said output signals in terms of products of different ones of said output signals, said parallel circuit comprising a plurality of amplifiers and a feedback conductance matrix connecting the output of each of said amplifiers,  $V_i$ , to the input of each of said amplifiers, with each of said connections being effected through a conductance  $T_{ijkl}$ ; where said feedback matrix comprises conductances  $T_{ijkl}$  connecting the output of amplifier  $ij$  to the input of amplifier  $kl$  with conductance values which are linear combinations of  $\lambda_x$  and  $\delta_z^y$  terms, where  $\lambda_x$  are predetermined penalty constants and  $\delta_z^y$  is a function whose value is one when  $y=z$ , and zero otherwise, where  $y$  and  $z$  belong to the set



{i,j,k,l}, the values of said conductances being computed in accordance with the set of problem restrictions, CHARACTERISED IN THAT said terms of products of different ones of said output signals comprise product terms of degree greater than two, said plurality of amplifiers comprises M amplifiers where M is greater than N by the number of terms in said energy function whose order is greater than two, thus forming a set of N amplifiers that provide output and a set of M-N amplifiers (16,17,18) where each amplifier in said M-N amplifier set corresponds to a term in said energy function whose order is greater than two; wherein for each term in said energy function whose degree is greater than two, said feedback matrix is augmented by the coefficients of the expression  $V_u V_m + \lambda [V_m + (-1 + V_w + V_x + \dots)]^2$  where  $V_u$ ,  $V_w$  and  $V_x$  are said amplifier output voltages included in that term of said energy function whose degree is greater than two, and  $V_m$  is the amplifier output voltage of an additional amplifier included in said set of M-N amplifiers.

2. A circuit as claimed in claim 1 comprising an input matrix for injecting a predetermined current into each of said inputs of said amplifiers.
3. A circuit as claimed in claim 2 wherein said input matrix is a combination of preselected input signals and a linear combination of  $\lambda_x$  and  $\delta_x^y$  terms, said preselected input signals uniquely characterise the circuit's response to stimuli,  $\lambda_x$  are preselected penalty constants, and  $\delta_x^y$  is a function in y and z whose value is one when  $y=z$  and zero otherwise, with y and z belonging to the set {i,j,k,l}.
4. A circuit as claimed in claim 1 wherein for each term in said energy function of the form

$$V_{AB} \prod_C \bar{V}_{CA},$$

said plurality of M amplifiers is increased by one amplifier developing an output voltage  $V_{inA}$  and said feedback matrix is augmented with a term of the form

$$\sum_{i,j,k,l} | \delta_i^j \delta_k^{in} + \lambda \delta_i^{in} \delta_k^j + 2\lambda \delta_i^{in} \delta_j^k + \lambda \delta_i^j |$$

where  $\lambda$  is a preselected penalty constant, and  $\delta$  is a function in the variables appearing as a subscript and a superscript of  $\delta$ , whose value is one when subscript is equal to the superscript and zero otherwise.

5. A circuit as claimed in claim 1 wherein for each term in said energy function of the form

$$V_{AB} \prod_D \bar{V}_{CD},$$

said plurality of M amplifiers is increased by one amplifier developing an output voltage  $V_{Cout}$  and said feedback matrix is augmented with a term of the form

$$\sum_{i,j,k,l} | \delta_k^i \delta_l^{out} + \lambda \delta_k^i \delta_l^j + 2\lambda \delta_k^i \delta_j^l + \lambda \delta_k^i |$$

where  $\lambda$  is a preselected penalty constant, and  $\delta$  is a function in the variables appearing as a subscript and a superscript of  $\delta$ , whose value is one when subscript is equal to the superscript and zero otherwise.

## Patentansprüche

1. Parallele Schaltung für das Entwickeln N simultaner Ausgangssignale, die eine Energiefunktion minimieren, welche durch die Ausgangssignale in Termen von Produkten von verschiedenen der Ausgangs-



signale ausgedrückt wird, wobei die Parallelschaltung eine Vielzahl von Verstärkern und eine Rückkopplungsleitwertmatrix, die den Ausgang jedes Verstärkers,  $V_{ij}$ , mit dem Eingang jedes der Verstärker verbindet, enthält, wobei jede der Verbindungen durch eine Konduktanz  $T_{ijk}$  bewirkt wird, wobei die Rückkopplungsmatrix Konduktanzen  $T_{ijk}$  enthält, die den Ausgang des Verstärkers  $ij$  mit dem Eingang des Verstärkers  $kl$  mit Konduktanzwerten verbinden, die Linearkombinationen von  $\lambda_x$  und  $\delta_z^y$  Termen sind, wobei  $\lambda_x$  eine vorbestimmte Strafkonstante und  $\delta_z^y$  eine Funktion ist, deren Wert eins ist, wenn  $y=z$  und andernfalls null ist, wobei  $y$  und  $z$  zu dem Satz  $\{ijk\}$  gehören, wobei die Werte der Konduktanzen gemäß dem Satz an Problemrestriktionen berechnet ist, dadurch gekennzeichnet,

daß die Terme aus Produkten verschiedener Ausgangssignale Produktterme einer Ordnung größer als 2 enthalten, die Vielzahl an Verstärkern  $M$  Verstärker enthält, wobei  $M$  um die Anzahl an Termen in der Energiefunktion, deren Ordnung höher als 2 ist, größer als  $N$  ist, wodurch ein Satz aus  $N$  Verstärkern gebildet wird, die ein Ausgangssignal bereitstellen und ein Satz aus  $M-N$  Verstärkern (16, 17, 18), wobei jeder Verstärker des  $M-N$  Verstärkersatzes einem Term in der Energiefunktion entspricht, dessen Ordnung höher als 2 ist, wobei für jeden Term in der Energiefunktion, dessen Ordnung höher als zwei ist, die Rückkopplungsmatrix durch die Koeffizienten der Gleichung  $V_u V_m + \lambda[V_m + (-1 + \overline{V_w} + \overline{V_x} + \dots)]^2$  erweitert ist, wobei  $V_u$ ,  $V_w$  und  $V_x$  die Verstärkerausgangsspannungen sind, die in diesem Term der Energiefunktion, dessen Ordnung höher als 2 ist, enthalten sind, und  $V_m$  die Verstärkerausgangsspannung des zusätzlichen in dem Satz aus  $M-N$  Verstärkern enthaltenen Verstärkers ist.

2. Schaltung nach Anspruch 1 mit einer Eingangsmatrix für das Injizieren eines vorbestimmten Stromes in jeden Eingang der Verstärker.

3. Schaltung nach Anspruch 2, in welcher die Eingangsmatrix eine Kombination aus vorausgewählten Eingangssignalen und eine Linearkombination aus  $\lambda_x$  und  $\delta_x^y$  Termen ist, wobei die vorausgewählten Eingangssignale die Schaltungsantwort auf Stimulationen in eindeutiger Weise beschreiben,  $\lambda_x$  vorausgewählte Strafkonstanten sind und  $\delta_z^y$  eine Funktion von  $y$  und  $z$  ist, deren Wert eins ist, wenn  $y=z$  und andernfalls null ist, wobei  $y$  und  $z$  zu dem Satz  $\{i,j,k,l\}$  gehören.

4. Schaltung nach Anspruch 1, in welcher für jeden Term in der Energiefunktion der Form

$$V_{AB} \prod_C \overline{V_{CA}}$$

die Vielzahl von  $M$  Verstärkern um einen Verstärker erhöht wird, der eine Ausgangsspannung  $V_{inA}$  entwickelt und die Rückkopplungsmatrix um einen Term der Form

$$\sum_{i,j,k,l} |\delta_i^j \delta_k^m + \lambda \delta_i^m \delta_k^j \delta_l^j + 2\lambda \delta_i^m \delta_l^j + \lambda \delta_l^j|$$

erweitert wird, wobei  $\lambda$  eine vorausgewählte Strafkonstante ist und  $\delta$  eine Funktion der als tiefgestellter Index und als hochgestellter Index von  $\delta$  auftretenden Variablen ist, deren Wert eins ist, wenn die tiefgestellten Indizes gleich den hochgestellten Indizes ist und ansonsten null ist.

5. Schaltung nach Anspruch 1, in welcher für jeden Term in der Energiefunktion der Form

$$V_{AB} \prod_D \overline{V_{CD}}$$

die Vielzahl aus  $M$  Verstärkern um einen Verstärker erhöht wird, der eine Ausgangsspannung  $V_{Cout}$  entwickelt und die Rückkopplungsmatrix mit einem Term der Form

$$\sum_{i,j,k,l} |\delta_i^j \delta_k^m + \lambda \delta_i^m \delta_k^j \delta_l^j + 2\lambda \delta_i^m \delta_l^j + \lambda \delta_l^j|$$



erweitert wird, wobei  $\lambda$  eine vorausgewählte Strafkonstante ist und  $\delta$  eine Funktion der Variablen ist, die als tiefgestellter und als hochgestellter Index von  $\delta$  auftreten, deren Wert eins ist, wenn die tiefgestellten Indizes gleich den hochgestellten Indizes sind und ansonsten null ist.

## 5 Revendications

1. Un circuit de calcul parallèle qui est destiné à développer N signaux de sortie simultanés qui minimisent une fonction d'énergie qui est exprimée par ces signaux de sortie à l'aide de termes de produit de différents signaux parmi ces signaux de sortie, ce circuit parallèle comprenant un ensemble d'amplificateurs et une matrice de conductances de rétroaction qui connecte la sortie de chacun des amplificateurs,  $V_{ij}$ , à l'entrée de chacun des amplificateurs, chacune de ces connexions étant effectuée par l'intermédiaire d'une conductance  $T_{ijk}$ ; la matrice de rétroaction comprenant des conductances  $T_{ijk}$  qui connectent la sortie d'amplificateur  $ij$  à l'entrée d'amplificateur  $kl$ , avec des valeurs de conductances qui sont des combinaisons linéaires de termes  $\lambda_x$  et  $\delta_z^y$ , dans lesquels les  $\lambda_x$  sont des constantes de pénalité prédéterminées et  $\delta_z^y$  est une fonction dont la valeur est égale à un lorsque  $y = z$ , et à zéro dans le cas contraire,  $y$  et  $z$  appartenant à l'ensemble  $\{i, j, k, l\}$ , les valeurs de ces conductances étant calculées conformément à l'ensemble de restrictions du problème, CARACTERISE EN CE QUE les termes de produits de différents signaux parmi les signaux de sortie comprennent des termes de produit de degré supérieur à deux, l'ensemble d'amplificateurs comprend M amplificateurs, M étant supérieur à N avec un écart égal au nombre de termes dans la fonction d'énergie dont l'ordre est supérieur à deux, ce qui force un ensemble de N amplificateurs fournissant un signal de sortie, et un ensemble de M-N amplificateurs (16, 17, 18), chaque amplificateur dans cet ensemble de M-N amplificateurs correspondant à un terme dans la fonction d'énergie dont l'ordre est supérieur à deux; et pour chaque terme dans la fonction d'énergie dont le degré est supérieur à deux, la matrice de rétroaction est augmentée des coefficients de l'expression  $V_u V_m + \lambda [V_m + (-1 + V_w + V_x + \dots)]^2$ , dans laquelle  $V_u$ ,  $X_w$  et  $V_x$  sont des tensions de sortie d'amplificateurs qui sont incluses dans le terme considéré de la fonction d'énergie dont le degré est supérieur à deux, et  $V_m$  est la tension de sortie d'amplificateur d'un amplificateur supplémentaire qui est inclus dans l'ensemble précité de M-N amplificateurs.
2. Un circuit selon la revendication 1, comprenant une matrice d'entrée pour injecter un courant prédéterminé dans chacune des entrées des amplificateurs.
3. Un circuit selon la revendication 2, dans lequel la matrice d'entrée est une combinaison de signaux d'entrée présélectionnés et une combinaison linéaire de termes  $\lambda_x$  et  $\delta_z^y$ ; ces signaux d'entrée présélectionnés caractérisent de façon spécifique la réponse du circuit à des stimulus,  $\lambda_x$  sont des constantes de pénalité présélectionnées, et  $\delta_z^y$  est une fonction en  $y$  et  $z$  dont la valeur est égale à un lorsque  $y = z$  et à zéro dans le cas contraire,  $y$  et  $z$  appartenant à l'ensemble  $\{i, j, k, l\}$ .
4. Un circuit selon la revendication 1, dans lequel pour chaque terme dans la fonction d'énergie de la forme

$$V_{AB} \prod_C \bar{V}_{CA}$$

l'ensemble de M amplificateurs est augmenté d'un amplificateur qui produit une tension de sortie  $V_{inA}$ , et la matrice de rétroaction est augmentée d'un terme de la forme

$$\sum_{i, j, k, l} | \delta_1^i \delta_k^{in} + \lambda \delta_i^{in} \delta_k^i \delta_{1+2}^i + 2\lambda \delta_i^{in} \delta_{1+}^i + \lambda \delta_1^i |.$$

dans lequel  $\lambda$  est une constante de pénalité présélectionnée et  $\delta$  est une fonction des variables apparaissant sous la forme d'un indice inférieur et d'un indice supérieur de  $\delta$ , dont la valeur est égale à un lorsque l'indice inférieur est égal à l'indice supérieur, et à zéro dans le cas contraire.



5. Un circuit selon la revendication 1, dans lequel pour chaque terme dans la fonction d'énergie de la forme

$$V_{AB} \prod_D \bar{V}_{CD},$$

l'ensemble de M amplificateurs est augmenté d'un amplificateur qui produit une tension de sortie  $V_{\text{out}}$ , et la matrice de rétroaction est augmentée d'un terme de la forme

$$\sum_{i,j,k,l} | \delta_k^j \delta_l^{\text{out}} + \lambda \delta_k^i \delta_l^j \delta_j^i + 2\lambda \delta_k^i \delta_j^i + \lambda \delta_k^i |,$$

dans lequel  $\lambda$  est une constante de pénalité présélectionnée et  $\delta$  est une fonction des variables apparaissant sous la forme d'un indice inférieur et d'un indice supérieur de  $\delta$ , dont la valeur est égale à un lorsque l'indice inférieur est égal à l'indice supérieur, et à zéro dans le cas contraire.